REGRESSION ANALYSIS WITH CORRELATED OBSERVATIONS

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THESIS

REGRESSION ANALYSIS
WITH
CORRELATED OBSERVATIONS

by

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The regression model $\underline{Y} = \underline{XB} + \underline{e}$, with $\underline{e} \sim N(\underline{0}, \sigma^2 \underline{I})$, has been studied extensively. That is, the model in which the errors are independent and identically distributed as $N(0, \sigma^2)$ has been studied already.

In this thesis we study the model in which the sample observations are correlated with a prescribed correlation structure and show that many of the results available for the independent case apply equally well for the correlated samples.

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Regression Analysis
with
Correlated Observations

by

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In this thesis we study the model in which the sample observations are correlated with a prescribed correlation structure and show that many of the results available for the independent case apply equally well for the correlated samples.

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TABLE OF CONTENTS

I. INTRODUCTION

Regression analysis is designed for situations where a variable is thought to be related to one or more other measurements made, usually, on the same objects. A purpose of the analysis is to use data (observed values of the variables) to estimate the form of this relationship.

The problem we consider is that of estimating or predicting the value of a dependent random variable y on the basis of some known measurements of an independent controlled variable x. Scientists, economists, psychologists, and sociologists have always been concerned with the problems of prediction. Meterologists are constantly analyzing data in hopes of predicting or forecasting with a high degree of accuracy.

An example would be to use information on weight and height of people to estimate the extent to which a man's weight is related to his height. If among people one was picked at random we might expect his weight to be a + bx (where x is height), namely, E(y) = a + bx (where y is weight). In gathering data, the weight of every man with height x will not be exactly a + bx. Therefore the difference, $y_i - E(y_i)$, can be written as

$$e_i = y_i - E(y_i) = y_i - a - bx_i \ ,$$

$$\text{hence } y_i = a + bx_i + e_i \ .$$

The man's age can also be considered to be a factor affecting weight. Then the model is extended to be

$$E(y) = a + b_1x_1 + b_2x_2 .$$

More generally, we can extend the equation to be

$$E(y) = a + b_1x_1 + b_2x_2 + \cdots + b_kx_k$$

$$y_i = a + b_1x_1 + b_2x_2 + \cdots + b_kx_k + e_i .$$

A frequently used notation is

$$y_i = b_0x_{i0} + b_1x_{i1} + b_2x_{i2} + \cdots + b_kx_{ik} + e_i ,$$

for $i = 1,2,\cdots,n$, with $x_{i0} = 1$ for all i.

Now define the following matrices and vectors:

$$\underline{X} = \begin{bmatrix} x_{10} & x_{11} & \cdots & x_{1k} \\ x_{20} & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & & \vdots \\ x_{n0} & x_{n1} & \cdots & x_{nk} \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n1} & \cdots & x_{nk} \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix}, \quad \underline{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}.$$

Then we write

$$\underline{Y} = \underline{XB} + \underline{e} , \text{ with } E(\underline{Y}) = \underline{XB} .$$

The model $\underline{Y}=\underline{XB} + \underline{e}$, with $\underline{e} \sim N(\underline{0}, \sigma^2\underline{I})$, has been studied extensively. That is, the model in which the errors are independent and identically distributed as $N(0, \sigma^2)$ has been studied already.

In this thesis we study the model in which the sample observations are correlated with a prescribed correlation structure and show that many of the results available for the independent case apply equally well for the correlated samples.

In this thesis, we shall examine the model

$$\underline{Y} = \underline{XB} + \underline{e} \quad , \text{ with } \underline{e} \sim N(\underline{0}, \underline{V}) \quad , \text{ where}$$

$$\underline{V} = \tfrac{1}{2}(\underline{A} + \underline{A}') + \alpha (\underline{I} - \underline{E}) ,$$

$$\underline{A} = \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix} ,$$

$\underline{A}'$ is the transpose of $\underline{A}$, a_i($i = 1,2,\cdots,n$) and α are positive constants, $\underline{I}$ is an n x n identity matrix, and $\underline{E}$ is an n x n matrix all of whose elements are unity.

We shall find that some results obtained here are not just the same as the case where the errors are independent and identically distributed as $N(0, \sigma^2)$.

II. SUMMARY OF KNOWN RESULTS

A. SOME PROPERTIES OF $\hat{\underline{B}}$ AND $\hat{\sigma}^2$ UNDER NORMAL THEORY

The model $\underline{Y} = \underline{XB} + \underline{e}$, $\underline{e} \sim N(\underline{0}, \sigma^2 \underline{I})$, where the components of $\underline{B}$ and σ^2 are unknown, and $\underline{X}$ is a known matrix of rank $k + 1$. Since the vector of errors $\underline{e}$ is normally distributed, the maximum-likelihood method will be used to estimate $\underline{B}$ and σ^2. Solving the likelihood equation we get

$$\hat{\underline{B}} = \begin{bmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \vdots \\ \hat{b}_k \end{bmatrix} = \underline{S}^{-1}\underline{X}'\underline{Y}\,, \quad \text{where } \underline{S}^{-1} = (\,\underline{X}'\underline{X}\,)^{-1}\,,$$

and

$$\hat{\sigma}^2 = \frac{(\,\underline{Y} - \underline{X\hat{B}}\,)'\,(\,\underline{Y} - \underline{X\hat{B}}\,)}{n - (\,k + 1\,)} = \frac{\underline{Y}'\,(\,\underline{I} - \underline{XS}^{-1}\underline{X}'\,)\,\underline{Y}}{n - (\,k + 1\,)}\,.$$

Then, $\hat{\underline{B}}$ and $\hat{\sigma}^2$ have the following properties:

1. consistent
2. efficient
3. unbiased
4. sufficient
5. $\hat{\underline{B}} \sim N(\,\underline{B}\,, \sigma^2 \underline{S}^{-1}\,)$
6. complete
7. minimum variance unbiased
8. $\dfrac{(\,n - k - 1\,)\,\hat{\sigma}^2}{\sigma^2} \sim \chi^2(\,n - k - 1\,)$
9. $\hat{\underline{B}}$ and $\hat{\sigma}^2$ are independent.

B. SOME THEOREMS USED IN DERIVING THE RESULTS

Theorem 1. If $\underline{Y}$ is distributed as $N(\underline{\mu}, \sigma^2 \underline{I})$, then $\frac{\underline{Y}'\underline{AY}}{\sigma^2}$ is distributed as $\chi^2(k, \lambda)$, where $\lambda = \frac{\underline{\mu}' \underline{A} \underline{\mu}}{2\sigma^2}$ and k = rank of $\underline{A}$, if and only if $\underline{A}$ is idempotent.

Theorem 2. If $\underline{Y}$ is distributed as $N(\underline{\mu}, \underline{V})$, then $\underline{Y}'\underline{BY}$ is distributed as $\chi^2(k, \lambda)$, where $\lambda = \frac{1}{2}\underline{\mu}'\underline{B}\underline{\mu}$ and k = rank of $\underline{B}$, if and only if $\underline{BV}$ is idempotent.

Theorem 3. If $\underline{Y} \sim N(\underline{\mu}, \underline{V})$, then $\underline{Y}'\underline{AY}$ and $\underline{Y}'\underline{BY}$ are independent if and only if $\underline{AVB} = \underline{0}$.

Theorem 4. If $\underline{Y} \sim N(\underline{\mu}, \underline{V})$, then $\underline{C}'\underline{Y}$ and $\underline{Y}'\underline{AY}$ are independent if and only if $\underline{C}'\underline{VA} = \underline{0}$.

Theorem 5. (Hogg and Craig theorem)

Let $Q = Q_1 + Q_2 + \cdots + Q_k$, where $Q, Q_1, \cdots, Q_{k-1}$ and Q_k are k + 1 random variables that are quadratic forms in the observations of a random sample of size n from $N(\mu, \sigma^2)$.

Let $\frac{Q}{\sigma^2}$ be $\chi^2(r)$, let $\frac{Q_i}{\sigma^2}$ be $\chi^2(r_i)$, $i = 1, 2, \cdots, k-1$, and let Q_k be non-negative. Then the random variables Q_1, Q_2, $\cdots$, Q_k are mutually stochastically independent and, hence, $\frac{Q_k}{\sigma^2}$ is $\chi^2(r_k = r - \sum_{i=1}^{k-1} r_i)$.

Theorem 6. (Baldessari theorem)

Let $\underline{Y}$ be $N(\underline{\mu}, \underline{V})$ and $\underline{B}_0$, $\underline{B}_1$, $\cdots$, $\underline{B}_k$ be (n x n) idempotent matrices satisfying

$$\sum_{j=0}^{k} \underline{B}_j = \underline{I} - \frac{1}{n}\underline{E}$$

, where $\underline{I}$ and $\underline{E}$ are matrices as defined above.

Let α be a positive constant.

Then a necessary and sufficient condition for $\underline{Y}'\underline{B}_j\underline{Y}/\alpha$, $j = 1,2,\cdots,k$, to be mutually independent and have non-central Chi-square distributions with r_j (r_j=rank of $\underline{B}_j$, $j = 1,2,\cdots,k$) degrees of freedom is that the covariance matrix $\underline{V}$ has the following structure

$$\underline{V} = \tfrac{1}{2}(\underline{A} + \underline{A}') + \alpha (\underline{I} - \underline{E}) .$$

III. SOME PROPERTIES OF $\hat{\underline{B}}$ AND $\underline{Y}'(\underline{I}-\underline{XS}^{-1}\underline{X}')\underline{Y}$ UNDER THE PRESENT MODEL

We consider the model

$$\underline{Y} = \underline{XB} + \underline{e}\ , \quad \underline{e}\sim N(\underline{0},\underline{V}),\ \text{where } \underline{V}=\tfrac{1}{2}(\underline{A}+\underline{A}')+\alpha(\underline{I}-\underline{E})\ .$$

Let $\hat{\underline{B}}=\underline{S}^{-1}\underline{X}'\underline{Y}$ and $\dfrac{\underline{Y}'\underline{CY}}{\alpha} = \dfrac{(\underline{Y}-\underline{X}\hat{\underline{B}})'(\underline{Y}-\underline{X}\hat{\underline{B}})}{\alpha} = \dfrac{\underline{Y}'(\underline{I}-\underline{XS}^{-1}\underline{X}')\underline{Y}}{\alpha}$

Now we will show that

A. $E(\hat{\underline{B}}) = \underline{B}$

B. $\dfrac{\underline{Y}'\underline{CY}}{\alpha} \sim \chi^2(n-k-1)$

C. $E(\underline{Y}'\underline{CY}) = \alpha\,(n-k-1)$ and

D. $\hat{\underline{B}}$ and $\underline{Y}'\underline{CY}$ are not independent.

A. EXPECTED VALUE OF $\hat{\underline{B}}$

$$E(\hat{\underline{B}}) = E(\underline{S}^{-1}\underline{X}'\underline{Y}) = \underline{S}^{-1}\underline{X}'E(\underline{Y}) = \underline{S}^{-1}\underline{X}'E(\underline{XB} + \underline{e}\ )$$
$$= \underline{B}$$

Hence $\hat{\underline{B}} = \underline{S}^{-1}\underline{X}'\underline{Y}$ is an unbiased estimate of $\underline{B}$

B. DISTRIBUTION OF $\dfrac{(\underline{Y}-\underline{X}\hat{\underline{B}})'(\underline{Y}-\underline{X}\hat{\underline{B}})}{\alpha}$

We have seen that $\dfrac{(n-k-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(n-k-1)$, where $\hat{\sigma}^2 = \dfrac{\underline{Y}'\underline{CY}}{n-k-1}$.

Therefore we can conjecture that $\dfrac{\underline{Y}'(\underline{I}-\underline{XS}^{-1}\underline{X}')\underline{Y}}{\alpha}$ is distributed as $\chi^2(n-k-1)$. The necessary and sufficient condition for $\dfrac{\underline{Y}'\underline{CY}}{\alpha}$ to be $\chi^2(n-k-1)$ is that $\frac{1}{\alpha}\,\underline{CV}$ is idempotent.

Now we want to show that $\frac{1}{\alpha}\,\underline{CV} = \frac{1}{\alpha}(\underline{I}-\underline{XS}^{-1}\underline{X}')\underline{V}$ is idempotent, that is, to show that $\frac{1}{\alpha}(\underline{I}-\underline{XS}^{-1}\underline{X}')\underline{V}(\underline{I}-\underline{XS}^{-1}\underline{X}') = \underline{I}-\underline{XS}^{-1}\underline{X}'$.

Note $\frac{1}{\alpha}(\underline{I}-\underline{XS}^{-1}\underline{X}')\underline{V}(\underline{I}-\underline{XS}^{-1}\underline{X}')$

$$=\frac{1}{\alpha}(\underline{I}-\underline{XS}^{-1}\underline{X}')\left(\tfrac{1}{2}(\underline{A}+\underline{A}')+\ (\underline{I}-\underline{E})\right)(\underline{I}-\underline{XS}^{-1}\underline{X}')$$

$$=(\underline{I}-\underline{XS}^{-1}\underline{X}')\ ,$$

since $\underline{XS}^{-1}\underline{X}'\underline{E}=\underline{E}$, $\underline{XS}^{-1}\underline{X}'\underline{A}'=\underline{A}'$ and $\underline{AXS}^{-1}\underline{X}'=\underline{A}$.

Hence, by theorem 2, $\frac{\underline{Y}'\underline{CY}}{\alpha} \sim \chi'^2(\ q\ ,\lambda\ )$,

where $q = \text{rank}\ \frac{(\ \underline{I}\ -\ \underline{XS}^{-1}\underline{X}'\ )}{\alpha}$

$$= \text{rank}\ (\ \underline{I}\ -\ \underline{XS}^{-1}\underline{X}'\ )$$

$$= \text{tr}\ (\ \underline{I}\ -\ \underline{XS}^{-1}\underline{X}'\ )$$

$$= \text{tr}\ (\ \underline{I}\ )\ -\ \text{tr}\ (\ \underline{X}'\underline{XS}^{-1}\ )$$

$$= n\ -\ k\ -\ 1$$

since $\underline{I}-\underline{XS}^{-1}\underline{X}'$ is itself symmetric and idempotent,

and $\lambda = \tfrac{1}{2}\underline{B}'\underline{X}'\frac{(\ \underline{I}\ -\ \underline{XS}^{-1}\underline{X}'\ )}{\alpha}\underline{XB} = 0$.

Therefore $\frac{\underline{Y}'\underline{CY}}{\alpha} \sim \chi^2\ (\ n-k-1\ )$.

C. EXPECTED VALUE OF $\underline{Y}'\underline{CY}$

From B it is obtained directly that $E(\underline{Y}'(\underline{I}-\underline{XS}^{-1}\underline{X}')\underline{Y})=$ $\alpha(n-k-1)$ also then $\frac{\underline{YCY}}{n-k-1}$ is an unbiased estimate of α.

D. $\hat{\underline{B}} = \underline{S}^{-1}\underline{X}'\underline{Y}$ AND $\underline{Y}'\underline{CY}$ ARE NOT INDEPENDENT

To show that they are not independent, let's show

$\underline{S}^{-1}\underline{X}'\underline{VC} = \underline{S}^{-1}\underline{X}'\underline{V}(\underline{I}-\underline{XS}^{-1}\underline{X}') \neq \underline{0}$ (Th. 4) .

$$\underline{S}^{-1}\underline{X}'\underline{V}(\underline{I}-\underline{XS}^{-1}\underline{X}')=\underline{S}^{-1}\underline{X}'\left(\tfrac{1}{2}(\underline{A}+\underline{A}')+\alpha(\underline{I}-\underline{E})\right)(\underline{I}-\underline{XS}^{-1}\underline{X}')$$

$$=\tfrac{1}{2}(\underline{S}^{-1}\underline{X}'\underline{A}'-\underline{S}^{-1}\underline{X}'\underline{A}'\underline{XS}^{-1}\underline{X}') \neq \underline{0}\ .$$

We recognize that $\hat{\underline{B}}$ and $\underline{Y}'\underline{CY}$ are not independent, which is not the same results as obtained in the i.i.d. case.

For reference, the variance and covariance of $\hat{\underline{B}}$ is provided as follows:

$$V(\hat{\underline{B}})=V\begin{pmatrix}\hat{b}_0\\ \hat{\underline{B}}_1\end{pmatrix}$$

$$=\begin{bmatrix}\bar{a}-\alpha(n-1)/n+(n\bar{a}\underline{\bar{x}}'-\underline{a}'\underline{X}_1+\alpha\underline{\bar{x}}')\underline{S}^{-1}\underline{\bar{x}} & (\tfrac{1}{2}\underline{a}'\underline{X}_1-\tfrac{1}{2}n\bar{a}\underline{\bar{x}}'-\alpha\underline{\bar{x}}')\underline{S}_1^{-1}\\ \underline{S}_1^{-1}(\tfrac{1}{2}\underline{X}_1'\underline{a}-\tfrac{1}{2}n\bar{a}\underline{\bar{x}}-\alpha\underline{\bar{x}}) & \alpha\,\underline{S}_1^{-1}\end{bmatrix},$$

where $\bar{a} = (1/n)\sum_{i=1}^{n} a_i$, $\underline{a}' = (\ a_1\ a_2\ \cdots\ a_n\ )$,

$\underline{\bar{x}}' = (\ \bar{x}_{\cdot 1}\ \bar{x}_{\cdot 2} \cdots \bar{x}_{\cdot k}\ )$, $\bar{x}_{\cdot j}=(1/n)\sum_{i=1}^{n} x_{ij}$ for all j ,

$$\underline{X}_1 = \begin{bmatrix} x_{11} & \cdots & x_{1k}\\ \vdots & & \vdots\\ x_{n1} & \cdots & x_{nk}\end{bmatrix}$$

and

$$\underline{S}_1 = \underline{X}_1'\underline{X}_1 - n\underline{\bar{x}}\underline{\bar{x}}' \quad .$$

Note that α factors from the lower right-hand corner, but not from the rest of the matrix. This causes trouble because the unknown parameter α does not cancel out in forming the usual Student t statistic.

Therefore we develop the new modified model without b_0 in the next chapter.

IV. SOME PROPERTIES OF $\hat{\underline{B}}_1$ AND $\underline{Y}^{*\prime}(\,\underline{I} - \underline{X}^*\underline{S}^{*-1}\underline{X}^{*\prime})\underline{Y}^*$ UNDER THE NEW MODEL

We consider the model

$$\underline{Y}^* = \underline{X}^*\underline{B}_1 + \underline{e}^* ,$$

where

$$\underline{Y}^* = \underline{Y} - \bar{y}\underline{1} = (\,\underline{I} - (1/n)\underline{E}\,)\underline{Y}$$

$$\underline{X}^* = \underline{X}_1 - (\bar{x}_{\cdot 1}\underline{1} \,\vdots\, \cdots \,\vdots\, \bar{x}_{\cdot k}\underline{1}) = (\,\underline{I} - (1/n)\underline{E}\,)\underline{X}_1$$

$$\underline{B}_1 = (\,b_1\ b_2\ \cdots\ b_k\,)'$$

$$\underline{e}^* = \underline{e} - \bar{e}\underline{1} = (\,\underline{I} - (1/n)\underline{E}\,)\underline{e} \ .$$

Note then that $y_i^* = y_i - \bar{y}$, $x_{ki}^* = x_{ki} - \bar{x}_{\cdot i}$ and thus $\sum_{i=1}^{n} y_i^* = \sum_{k=1}^{n} x_{ki}^* = 0$, that is, $\underline{1}'\underline{Y}^* = 0$, $\underline{1}'\underline{X}^* = \underline{0}$.

Then

$$V(\underline{e}^*) = \alpha(\,\underline{I} - (1/n)\underline{E}\,) ,$$

$$\text{i.e.,}\quad \underline{e}^* \sim N(\,\underline{0}\ ,\ \alpha(\underline{I} - (1/n)\underline{E}\,)\,)$$

and $$\underline{Y}^* \sim N(\,\underline{X}^*\underline{B}_1\ ,\ \alpha(\,\underline{I} - (1/n)\underline{E}\,)\,)$$

under our original assumptions.

Let $\hat{\underline{B}}_1 = \underline{S}^{*-1}\underline{X}^{*\prime}\underline{Y}^*$, where $\underline{S}^* = \underline{X}^{*\prime}\underline{X}^*$, and

$$\frac{\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*}{\alpha} = \frac{\underline{Y}^{*\prime}(\underline{I} - \underline{X}^*\underline{S}^{*-1}\underline{X}^{*\prime})\underline{Y}^*}{\alpha} = \frac{(\underline{Y}^* - \underline{X}^*\underline{B}_1)'(\underline{Y}^* - \underline{X}^*\underline{B}_1)}{\alpha}$$

We are going to show

A. The expected value of $\hat{\underline{B}}_1$

B. The distribution of $\dfrac{\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*}{\alpha}$

C. The expected value of $\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*$

D. That $\hat{\underline{B}}_1$ and $\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*$ are independent.

A. EXPECTED VALUE OF $\hat{\underline{B}}_1$

$$E(\hat{\underline{B}}_1) = E(\underline{S}^{*-1}\underline{X}^{*'}\underline{Y}^{*})$$
$$= \underline{S}^{*-1}\underline{X}^{*'}E(\underline{Y}^{*})$$
$$= \underline{S}^{*-1}\underline{X}^{*'}E(\underline{X}^{*}\underline{B}_1+\underline{e}^{*}) = \underline{B}_1 \quad .$$

Hence $\hat{\underline{B}}_1=\underline{S}^{*-1}\underline{X}^{*'}\underline{Y}^{*}$ is an unbiased estimate of $\underline{B}_1$ it is easily shown that $\hat{\underline{B}}_1$ is the same as $\hat{\underline{B}}_1$ in $\hat{\underline{B}} = \begin{pmatrix} \hat{b}_0 \\ \hat{\underline{B}}_1 \end{pmatrix}$.

B. DISTRIBUTION OF $(1/\alpha)(\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*})$

Since $\underline{C}^{*}\alpha(\underline{I}-\frac{1}{n}\underline{E})=\underline{I}-\underline{X}^{*}\underline{S}^{*-1}\underline{X}^{*'}-\frac{1}{n}\underline{E}$ is idempotent, $(1/\alpha)\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*}\sim\chi^2(q,\lambda)$ by Th. 2, where

$$q = \text{rank } (1/\alpha)(\underline{I}-\underline{X}^{*}\underline{S}^{*-1}\underline{X}^{*'})$$
$$= \text{rank } (\underline{I}-\underline{X}^{*}\underline{S}^{*-1}\underline{X}^{*'})$$
$$= \text{tr } (\underline{I}-\underline{X}^{*}\underline{S}^{*-1}\underline{X}^{*'})$$
$$= \text{tr } (\underline{I}) - \text{tr } (\underline{X}^{*'}\underline{X}^{*}\underline{S}^{*-1})$$
$$= n - k$$

and $\lambda = \frac{1}{2}\underline{B}_1'\underline{X}^{*'}(1/\alpha)(\underline{I}-\underline{X}^{*}\underline{S}^{*-1}\underline{X}^{*'})\underline{X}^{*}\underline{B}_1 = 0$

Therefore $(1/\alpha)(\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*}) \sim \chi^2(n-k)$.

C. EXPECTED VALUE OF $\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*}$

It is obtained directly from B that $E(\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*})=\alpha(n-k)$. Also it is easily seen that $\frac{\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*}}{n-k}$ is an unbiased estimate of α.

D. $\hat{\underline{B}}_1 = \underline{S}^{*-1}\underline{X}^{*\prime}\underline{Y}^{*}$ AND $\underline{Y}^{*\prime}\underline{C}^{*}\underline{Y}^{*}$ ARE INDEPENDENT

$$\underline{S}^{*-1}\underline{X}^{*\prime}\alpha(\underline{I}-(1/n)\underline{E})(\underline{I}-\underline{X}^{*}\underline{S}^{*-1}\underline{X}^{*\prime})$$

$$= \alpha(\underline{S}^{*-1}\underline{X}^{*\prime}-(1/n)\underline{S}^{*-1}\underline{X}^{*\prime}\underline{E})(\underline{I}-\underline{X}^{*}\underline{S}^{*-1}\underline{X}^{*\prime})$$

$$= \underline{0} \qquad (\text{ note } \underline{X}^{*\prime}\underline{E} = \underline{0}) \quad .$$

Therefore, by Th. 4, they are independent.

V. INTERVAL ESTIMATION AND PREDICTION INTERVAL

The construction of interval estimators and a prediction interval to contain a single future observation in the i.i.d. case with a regression model has been discussed in many standard texts. In this chapter we extend these results to the case where the samples are correlated with a special type of covariance structure. We shall discuss

A. Covariance of $\hat{\underline{B}}_1$, $V(\hat{\underline{B}}_1)$

B. Interval estimation of $\underline{r}'\underline{B}_1$

C. Prediction interval for one future observation.

A. COVARIANCE OF $\hat{\underline{B}}_1$, $V(\hat{\underline{B}}_1)$

Since $\hat{\underline{B}}_1$ is equal to the product of a constant matrix $\underline{S}^{*-1}\underline{X}^{*'}$ and a normally distributed vector $\underline{Y}^*$, $\hat{\underline{B}}_1$ has the k-variate normal distribution.

We have already shown that the mean of $\hat{\underline{B}}_1$ is $\underline{B}_1$.

The covariance matrix of $\hat{\underline{B}}_1$ is

$$\begin{aligned}
V(\hat{\underline{B}}_1) &= E((\hat{\underline{B}}_1-\underline{B}_1)(\hat{\underline{B}}_1-\underline{B}_1)') \\
&= E((\underline{S}^{*-1}\underline{X}^{*'}\underline{Y}^*-\underline{B}_1)(\underline{S}^{*-1}\underline{X}^{*'}\underline{Y}^*-\underline{B}_1)') \\
&= E((\underline{S}^{*-1}\underline{X}^{*'}(\underline{X}^*\underline{B}_1+\underline{e}^*)-\underline{B}_1)(\underline{S}^{*-1}\underline{X}^{*'}(\underline{X}^*\underline{B}_1+\underline{e}^*)-\underline{B}_1)') \\
&= \underline{S}^{*-1}\underline{X}^{*'}V(\underline{e}^*)\underline{X}^*\underline{S}^{*-1} \\
&= \alpha\,\underline{S}^{*-1}\underline{X}^{*'}(\underline{I}-(1/n)\underline{E})\underline{X}^*\underline{S}^{*-1} \\
&= \alpha\,\underline{S}^{*-1}
\end{aligned}$$

So $\hat{\underline{B}}_1$ is distributed $N(\underline{B}_1 , \alpha\,\underline{S}^{*-1})$.

B. INTERVAL ESTIMATION OF $\underline{r}'\underline{B}_1$

Frequently, an experimenter is interested in setting confidence limits on some function of b_i. A method is available for one of the most frequently occurring cases, i.e., for the case of a linear function of the b_i.

We, here, discuss the case of a linear function of the b_i (except b_0). Let $\underline{r}$ be a known k x 1 vector of constants, then, to set confidence limits of size $1-\alpha_1$, on $\underline{rB}_1$, we proceed as follows.

Since $\underline{r}'\underline{B}_1$ is distributed $N(\underline{r}'\underline{B}_1, \alpha\underline{r}'\underline{S}^{*-1}\underline{r})$, it is clear that $\dfrac{\underline{r}'\hat{\underline{B}}_1-\underline{r}'\underline{B}_1}{(\alpha\underline{r}'\underline{S}^{*-1}\underline{r})^{\frac{1}{2}}}$ is distributed N(0,1). We also know $\dfrac{\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*}}{\alpha} \sim \chi^2(n-k)$ and independent of $\hat{\underline{B}}_1$.

Therefore, $\dfrac{\underline{r}'\hat{\underline{B}}_1-\underline{r}'\underline{B}_1}{(\alpha\underline{r}'\underline{S}^{*-1}\underline{r})^{\frac{1}{2}}}\left(\dfrac{(n-k)}{\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*}}\right)^{\frac{1}{2}} \sim t(n-k)$.

We arrive at the probability equation

$$P(\underline{r}'\hat{\underline{B}}_1 - ct_{\frac{\alpha_1}{2}} \leq \underline{r}'\underline{B}_1 \leq \underline{r}'\hat{\underline{B}}_1 + ct_{\frac{\alpha_1}{2}}) = 1-\alpha_1 ,$$

where $c = \left(\dfrac{(\underline{r}'\underline{S}^{*-1}\underline{r})\underline{Y}^{*'}\underline{C}^{*}\underline{Y}^{*}}{n-k}\right)^{\frac{1}{2}}$.

C. PREDICTION INTERVAL FOR ONE FUTURE OBSERVATION

Hahn [4] has derived prediction intervals to contain some future observations of i.i.d. random variables from a normal distribution for a regression model. In this section we extend this result to the case where the samples are correlated and have a special type of covariance structure to construct prediction interval to contain a single future observation.

Let y_{n+1} be the one future sample.

Define $\underline{Y}$ to be the (n+1) component vector including the original sample of size n and y_{n+1}. We assume $\underline{Y} \sim N(\underline{XB}, \underline{V})$, where $\underline{X}$ is $(n+1)x(k+1)$, of rank k+1 and $\underline{V}$ is as defined in III, and is $(n+1)x(n+1)$.

Define y^*_{n+1} as follows:

$$y^*_{n+1}=y_{n+1}-\bar{y}_{n+1} \quad , \text{ where } \quad \bar{y}_{n+1}=(1/(n+1))\sum_{i=1}^{n+1} y_i \quad .$$

Then, $y^*_{n+1}=y_{n+1}-(1/(n+1))(y_{n+1}+n\bar{y}_n) = (n/(n+1))(y_{n+1}-\bar{y}_n)$,

$$\text{where } y_n=(1/n)\sum_{i=1}^{n} y_i \ ,$$

and y^*_{n+1} can be expressed

$$y^*_{n+1}=(n/(n+1))(b_1x^*_{(n+1)1}+b_2x^*_{(n+1)2}+\cdots+b_kx^*_{(n+1)k}+e^*_{n+1})$$

$$=(n/(n+1))(\underline{B}_1'\underline{X}^*_{n+1}+\underline{e}^*_{n+1}) \ ,$$

where $\underline{X}^*_{n+1}=(x^*_{n+1\ 1}, x^*_{n+1\ 2}, \ldots, x^*_{n+1\ k})'$.

We also, therefore, know $E(y^*_{n+1})=n/(n+1)\underline{B}_1'\underline{X}^*_{n+1}$ and

$V(y^*_{n+1})=(n/(n+1))\alpha$ since

$$V(y^*_{n+1})=(n/(n+1))^2(V(y_{n+1})+V(\bar{y}_n)-2COV(y_{n+1},\bar{y}_n))$$

$$=(n/(n+1))^2(a_{n+1}+\bar{a}-((n-1)/n)\alpha-2(\tfrac{1}{2}\bar{a}+\tfrac{1}{2}a_{n+1}-\alpha))$$

$$=(n/(n+1))\alpha \ .$$

Next, let $\hat{y}^*_{n+1}=(n/(n+1))\hat{\underline{B}}_1'\underline{X}^*_{n+1}$

then $E(\hat{y}^*_{n+1})=(n/(n+1))\underline{B}_1'\underline{X}^*_{n+1}$ and

$V(\hat{y}^*_{n+1})=\alpha(n/(n+1))^2\underline{X}^{*\prime}_{n+1}\underline{S}^{*-1}\underline{X}^*_{n+1}$.

Now let $z = y^*_{n+1} - \hat{y}^*_{n+1}$,

then $E(z) = 0$

$$V(z) = \alpha(n/(n+1))(1+(n/(n+1))(\underline{X}^{*\prime}_{n+1}\underline{S}^{*-1}\underline{X}^*_{n+1})) .$$

Therefore, $z \sim N(\,0\,,\ \alpha(n/(n+1))(1+(n/(n+1))\underline{X}^{*\prime}_{n+1}\underline{S}^{*-1}\underline{X}^*_{n+1}))$.

The standardized variate $z' = z/(V(z))^{\frac{1}{2}}$ is normally distributed with mean 0 and variance 1. Furthermore, since $(1/\alpha)\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^* \sim \chi^2(n-k)$ and is independent of z',

the ratio

$$T = \frac{z'}{\left(\dfrac{\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*}{\alpha(n-k)}\right)^{\frac{1}{2}}} = (y^*_{n+1} - \hat{y}^*_{n+1})\left(\frac{n-k}{((n/n+1)+(n/n+1)^2\underline{X}^{*\prime}_{n+1}\underline{S}^{*-1}\underline{X}^*_{n+1})\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*}\right)^{\frac{1}{2}}$$

follows a Student t distribution with n-k degree of freedom.

Proof of independent relationship between $\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*$ and z.

$\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^*$ and $z = y^*_{n+1} - \hat{y}^*_{n+1}$ can be expressed in terms of $\underline{Y} = (y_1, \cdots, y_{n+1})'$ as follows:

$$\underline{Y}^{*\prime}\underline{C}^*\underline{Y}^* = \underline{Y}'\begin{pmatrix}\underline{I}-(1/n)\underline{E} \\ \underline{0}'\end{pmatrix}\underline{C}^*(\underline{I}-(1/n)\underline{E} \,\vdots\, \underline{0})\underline{Y} = \underline{Y}'\underline{D}'\underline{C}^*\underline{DY}$$

$$\begin{aligned} z &= y^*_{n+1} - \hat{y}^*_{n+1} \\ &= (n/n+1)((-\tfrac{1}{n}, \cdots -\tfrac{1}{n}, 1) - \underline{X}^{*\prime}_{n+1}\underline{S}^{*-1}\underline{X}^{*\prime}(\underline{I}-(1/n)\underline{E} \,\vdots\, \underline{0}))\underline{Y} \\ &= (n/(n+1))\underline{BY} \quad . \end{aligned}$$

Then it can be seen that $\underline{BVD}'\underline{C}^*\underline{D} = \underline{0}$, where $V = \frac{1}{2}(\underline{A}+\underline{A}') + \alpha(\underline{I}-\underline{E})$ is an (n+1)x(n+1) matrix.

Therefore, by Th. 4, they are independent.

Consequently, a two sided 100r% prediction interval to contain the future sample y^*_{n+1} using the regression obtained

from the n given observations is

$$y^*_{n+1} + t_{\frac{r}{2}} \left(\frac{((n/(n+1)) + (n/(n+1))^2 \underline{X}^{*\prime}_{n+1} \underline{S}^{*-1} \underline{X}^*_{n+1})(\underline{Y}^{*\prime} \underline{C}^* \underline{Y}^*)}{n - k} \right)^{\frac{1}{2}}$$

Finally, the lower limit, $\underline{L}$, and the upper limit, $\overline{U}$, for y_{n+1} are expressed as follows:

$$\underline{L} = ((n+1)/n)(y^*_{n+1} - t_{\frac{r}{2}} u) + \bar{y}_n$$

$$\overline{U} = ((n+1)/n)(y^*_{n+1} + t_{\frac{r}{2}} u) + \bar{y}_n ,$$

$$\text{where } u = \left(\frac{((n/(n+1)) + (n/(n+1)) \underline{X}^{*\prime}_{n+1} \underline{S}^{*-1} \underline{X}^*_{n+1})(\underline{Y}^{*\prime} \underline{C}^* \underline{Y}^*)}{n - k} \right)^{\frac{1}{2}} .$$

LIST OF REFERENCES

1. Baldessri, B., "Analysis of Variance of Dependent Data," Statistica, Vol. 26, p. 895-903, 1966.

2. Choi, C. H., Prediction Interval for Correlated Samples, Thesis, Naval Postgraduate School, 1974.

3. Graybill, Franklin A., An Introduction to Linear Statistical Models, Vol. 1, McGraw-Hill, New York.

4. Hahn, G. J., Prediction Intervals for a Normal Distribution, General Electric Company, TIS Report No. 71-C-038.

5. Hogg, R. V. and Craig, A. T., Introduction to Mathematical Statistics, 2nd. Edition, Macmillan, New York.

6. Larson, H. J., Introduction to Probability Theory and Statistical Inference, Wiley, New York .

7. Searle, S. R., Linear Models, Wiley, New York.

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